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ON THE SURJECTIVITY OF SOME TRACE MAPS

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ABSTRACT

Let K be a commutative ring with a unit element 1. Let Γ be a finite group acting on K via a map $t\colon \Gamma \to \mathrm{Aut}(K)$. For every subgroup $H \leq \Gamma$ define $\mathrm{tr}_H\colon K \to K^H$ by $\mathrm{tr}_H(x) = \sum_{\sigma \in H} \sigma(x)$. We prove

THEOREM: tr_Γ *is surjective onto* K^Γ *if and only if* tr_P *is surjective onto* K^P *for every (cyclic) prime order subgroup* P *of* Γ.

This is false for certain non-commutative rings K.

0. Introduction

Let K be a commutative ring with a unit element 1 and let Γ be a finite group acting on K via a morphism $t\colon \Gamma \to \mathrm{Aut}(K)$. For every subgroup H of Γ and $x \in K$ define the trace map $\mathrm{tr}_H\colon K \to K$ by

$$\mathrm{tr}_H(x) = \sum_{\sigma \in H} \sigma(x).$$

Clearly, the image of this map lies in K^H, the H invariant elements in K. In this paper I discuss the relation between the surjectivity of $\mathrm{tr}_\Gamma\colon K \to K^\Gamma$ and of $\mathrm{tr}_H\colon K \to K^H$ for various subgroups $H \leq \Gamma$. Note that tr_H is a K^H linear map, so that the surjectivity of tr_H onto K^H is equivalent to the existence of an element $x_H \in K$ with $\mathrm{tr}_H(x_H) = 1$. Using this, it is easily shown that the surjectivity of tr_Γ onto K^Γ implies the surjectivity of tr_H onto K^H for every subgroup H in Γ. (For, if $\{t_1, \ldots, t_n\}$ are representatives of the left cosets of H

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in Γ, then $x_H = t_1(x_\Gamma) + \cdots + t_n(x_\Gamma)$). In particular, if p is a prime number and $P \leq \Gamma$ a subgroup of order p, then tr_P is surjective onto K^P.

The main result is regarding the converse.

THEOREM 0.1: *Let Γ be a finite group acting on a commutative ring K. Let H be a subgroup of Γ with $\mathrm{tr}_H\colon K \to K^H$ surjective and let $I = \{g_1, \ldots, g_n\}$ be a set of representatives for the right cosets of H in Γ. Then the following conditions are equivalent;*

1. $\mathrm{tr}_\Gamma\colon K \to K^\Gamma$ *is surjective.*
2. *There exists $x \in K^H$ such that $\sum_{g_i \in I} g_i(x) = 1$.*
3. *For each subgroup $P \leq \Gamma$ of prime order that intersects H trivially, $\mathrm{tr}_P\colon K \to K^P$ is surjective.*
4. *For each subgroup $S \leq \Gamma$, $S \cap H = \langle 1 \rangle$, $\mathrm{tr}_S\colon K \to K^S$ is surjective.*

COROLLARY 0.2: *If $\mathrm{tr}_P\colon K \to K^P$ is surjective for all subgroups P of prime order, then $\mathrm{tr}_\Gamma\colon K \to K^\Gamma$ is surjective.*

Proof: Take $H = \langle 1 \rangle$. Now use implication (3) $\Rightarrow$ (1). ∎

The proof of Theorem 0.1 uses a tensor induction argument for skew group rings. The language used there is of relative projectiveness. Hence, in Proposition 1.2 we translate the surjectivity of the various trace maps into this language. The theorem is proved in §1.

Implication (4) $\Rightarrow$ (1) gives the following:

COROLLARY 0.3: *Let H be a subgroup of Γ. If there exists $x_H \in K$ with $\mathrm{tr}_H(x_H) = 1$ and for every $S \leq \Gamma$ such that $S \cap H = \langle 1 \rangle$ there exists $x_S \in K$ with $\mathrm{tr}_S(x_S) = 1$, then there exists $x_\Gamma \in K$ with $\mathrm{tr}_\Gamma(x_\Gamma) = 1$.*

In §2 we give an explicit formula for x_Γ given the elements x_H and x_S mentioned above.

Finally, in §3 we apply Theorem 0.1 to generalize the result obtained by Zhong ([Z], Cor. 5.4) on the global dimension of skew group rings $K_t\Gamma$, where K is a commutative Noetherian ring.

Remarks:

1. If the action is trivial, the surjectivity of tr_H is equivalent to ord(H) being invertible in K and the theorem reduces to a triviality.
2. If K is a field and the action of Γ is faithful, then the trace maps are surjective (the automorphisms are linearly independent).

3. Let k be a number field and let K be a finite Galois extension with Galois group $\Gamma = \text{Gal}\ (K/k)$. Let $\mathfrak{a}_K$, $\mathfrak{a}_k$ denote the rings of algebraic integers. The Galois group Γ acts on $\mathfrak{a}_K$ and $\mathfrak{a}_k$ is the fixed ring. The surjectivity of the trace map $\text{tr}_\Gamma : \mathfrak{a}_K \rightarrow \mathfrak{a}_k$ is equivalent to the extension K/k being tame (see [F], 3, Th.3).
4. ([NV]) Let $R = \oplus_{\sigma \in \Gamma} R_\sigma$ be a strongly graded ring by a finite group Γ and suppose that there exists an element $x_\Gamma \in Z(R_e)$ (e the identity element of Γ) such that $\text{tr}_\Gamma(x_\Gamma) = 1$. If R_e is left hereditary then R is left hereditary.
5. Let K and Γ be as in Theorem 0.1. Then $\hat{H}^0(\Gamma, K)$ is trivial if and only if

$$\coprod_{\substack{P \leq \Gamma \\ |P| = \text{prime}}} \hat{H}^0(P, K)$$

is trivial. Here $\hat{H}^*$ denotes the reduced homology. (See [Br], Chapter VI, Section 4.)
6. Corollary 0.2 is not true in general if one replaces K by a non-commutative ring R. Example: (This is essentially the case $p = 2$ of the example appearing in [HLS], p.184.)

Let $R = M_2(\mathbb{F}_2(x))$ be the ring of 2×2 matrices over the field $\mathbb{F}_2(x)$ (= rational functions over the field of two elements). Let $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle\sigma\rangle \times \langle\tau\rangle$ and define the following action on the ring R:

$$\sigma : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow \begin{pmatrix} d & cx \\ bx^{-1} & a \end{pmatrix},$$

$$\tau : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix},$$

where

$$a' = (x + 1)^{-1} [a + b + (c + d)x],$$
$$b' = (x + 1)^{-1} [(a + cx)x + (b + dx)],$$
$$c' = (x + 1)^{-1} [a + b + c + d],$$
$$d' = (x + 1)^{-1} [(a + c)x + (b + d)].$$

One checks that $\text{tr}_{\langle\sigma\rangle}$, $\text{tr}_{\langle\tau\rangle}$, $\text{tr}_{\langle\sigma\tau\rangle}$ are surjective onto

$$S = R^{\langle\sigma\rangle} = R^{\langle\tau\rangle} = R^{\langle\sigma\tau\rangle}$$

but $\text{tr}_\Gamma : R \rightarrow R^\Gamma$ is the zero map.

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1. Proof of the theorem

With the given action of Γ on K we construct the skew group ring $K_t\Gamma$. It is isomorphic to the group ring $K\Gamma$ as a left free module over K and the multiplication is defined by the rule

$$(x\sigma)(y\tau) = x\sigma(y)\sigma\tau, \quad x,y \in K, \quad \sigma,\tau \in \Gamma$$

where $\sigma(y) = t_\sigma(y)$ is the action of σ on y via the homomorphism t.

Definition 1.1: Let R be a ring and S a subring. We say that R is relative semisimple if every R-module M which is projective as an S-module is also projective as an R-module.

PROPOSITION 1.2: *Let $L = K_tH$ be a sub-skew group ring of $R = K_t\Gamma$ where H is a subgroup of Γ (in K_tH, t is restricted to H). Then an R-module M which is L projective is also R projective if and only if there exists $f \in \mathrm{End}_{K_tH}(M_H)$ with*

$$\sum_{g_i\in I} g_i(f) = \sum_{i\in I} g_i f g_i^{-1} = \mathrm{id}_M$$

where $I = \{g_1,\dots,g_n\}$ are representatives of the right cosets of H in Γ. (For group algebras this is in [CR], Th. 19.2.*)*

Proof: First note that if $f \in \mathrm{End}_{K_tH}(M)$, then $\sum_{g_i\in I} g_i f g_i^{-1}$ is K_tH-linear. Indeed,

$$\sum_{g_i\in I} g_i f g_i^{-1}(hm) = \sum_{g_i\in I} g_i f (h^{-1}g_i)^{-1}(m) = h\sum_{g_i\in I}(h^{-1}g_i)f(h^{-1}g_i)^{-1}(m)$$
$$=h\sum_{g_j\in I}(g_jh_j)f(g_jh_j)^{-1}(m) = h\sum_{g_j\in I} g_j f g_j^{-1}(m).$$

Now consider the epimorphism of $K_t\Gamma$ modules

$$(K_t\Gamma \otimes_{K_tH} K)\otimes_K M \to M,$$
$$(x\sigma\otimes y)\otimes m \to x\sigma(y)m,$$

where $K_t\Gamma \otimes_{K_tH} K$ has a left $K_t\Gamma$ structure via left multiplication on $K_t\Gamma$ and $(K_t\Gamma \otimes_{K_tH} K)\otimes_K M$ has the left $K_t\Gamma$ diagonal structure (see [A], §1 for details).

Now the splitting of this map over $K_t\Gamma$ is equivalent to the existence of a map $f \in \mathrm{End}_{K_tH}(M)$ with $\sum_{g_i\in I} g_i f g_i^{-1} = \mathrm{id}_M$. So, the result will follow if we show that M being K_tH-projective implies $(K_t\Gamma \otimes_{K_tH} K) \otimes_K M$ is $K_t\Gamma$ projective (with the given action). Indeed, there is a natural isomorphism of functors

$$\begin{aligned} F &= \mathrm{Hom}_{K_t\Gamma}\Big((K_t\Gamma \otimes_{K_tH} K\big) \otimes_K M, -\Big) \\ &\simeq \mathrm{Hom}_{K_t\Gamma}\Big(K_t\Gamma, \mathrm{Hom}_{K_tH}(M,-)\Big) \simeq \mathrm{Hom}_{K_tH}(M,-). \end{aligned}$$

Hence, F is exact provided that M is K_tH-projective.

COROLLARY 1.3: *The trace map* $\mathrm{tr}_\Gamma\colon K \to K^\Gamma$ *is surjective if and only if* K *is projective as a* $K_t\Gamma$*-module.*

Proof: Take $M = K$ and $H = \langle 1\rangle$. Now apply the proposition. ∎

The equivalence of conditions (1) and (2) in Theorem 0.1 follows immediately from Proposition 1.2 for $M = K$ and Corollary 1.3.

The implications (1) $\Rightarrow$ (4) $\Rightarrow$ (3) are clear (see the introduction for (1) $\Rightarrow$ (4)).

To show (3) $\Rightarrow$ (4) let $S_0 \le \Gamma$ be a minimal counter example, i.e., $S_0 \cap H = \langle 1\rangle$, $\mathrm{tr}_{S_0}\colon K \to K^{S_0}$ not surjective and for every proper subgroup $N < S_0$

$$\mathrm{tr}_N\colon K \to K^N \text{ is surjective.}$$

Clearly, S_0 is not of prime order, so it contains a proper subgroup $N \neq \langle 1\rangle$. We know that $\mathrm{tr}_N\colon K \to K^N$ is surjective and every subgroup $T \le S_0$, $T \cap N = \langle 1\rangle$ is also proper, so that $\mathrm{tr}_T\colon K \to K^T$ is surjective. Hence, the implication (4) $\Rightarrow$(1) (shown below) for S_0 gives a contradiction to the minimality of S_0.

To show (4) $\Rightarrow$(1), assume (4) and that $\mathrm{tr}_H\colon K \to K^H$ is surjective. We show that $\mathrm{tr}_\Gamma\colon K \to K^\Gamma$ is surjective. Consider the short exact sequence of K_tH-modules.

$$0 \to J \to K_tH \xrightarrow{\epsilon} K \to 0$$

where $\epsilon\big(\sum_{\sigma\in H} x_\sigma\sigma\big) = \sum_{\sigma\in H} x_\sigma$ and $J = \mathrm{Ker}\,\epsilon$.

Since $\mathrm{tr}_H\colon K \to K^H$ is surjective, the map ϵ splits over K_tH and there exists an element $u = \sum_{h\in H} h(x_H)h$ with $\sum_{h\in H} h(x_H) = 1$. Clearly, $K_tH \cong J \oplus Ku$.

Now let

$$\begin{aligned} F = K_tH \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} K_tH/\big\{ &\operatorname*{span}_{\mathbb{Z}}\big(h_1 \otimes \cdots \otimes xh_i \otimes \cdots \otimes h_n - \\ &h_1 \otimes \cdots \otimes g_j^{-1}g_i(x)h_j \otimes \cdots \otimes h_n\big) \text{ all } x \in K,\ 1 \le i,\ j \le n\big\} \end{aligned}$$

where $I = \{g_1 = 1,\ g_2, \dots, g_n\}$ is a right transversal.

Define an action of $K_t\Gamma$ on F as follows (see [A], p.168). For $\sigma \in \Gamma$, $g_i \in I$ define $\nu_i \in \mathrm{Sym}(n)$, $g_{\nu_i} \in I$, $h_{\nu_i} \in H$ by the formula

$$(1.3.1) \qquad \sigma^{-1} g_i = g_{\nu_i} h_{\nu_i}^{-1}.$$

Then

$$x\sigma(z_1 \otimes \cdots \otimes z_n) = h_{\nu_1} z_{\nu_1} \otimes \cdots \otimes g_j^{-1}(x) h_{\nu_j} z_{\nu_j} \otimes \cdots \otimes h_{\nu_n} z_{\nu_n}.$$

Now let $L = K(\overline{u \otimes \cdots \otimes u})$ be the K submodule generated by $u \otimes \cdots \otimes u$ in F. Since u is H invariant, the K-module L is also a $K_t\Gamma$-module. Hence, the decomposition of K_tH modules $K_tH \simeq J \oplus Ku$ induces the decomposition of $K_t\Gamma$ modules $F = L \oplus E$ where E is the image in F of

$$\sum P^{\epsilon_1} \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} P^{\epsilon_n},$$
$$(\varepsilon_1 \dots \varepsilon_n) \neq (0, \dots, 0),$$
$$\varepsilon_i = \begin{cases} 0 \\ 1 \end{cases} \quad P^0 = \mathrm{Ku},\ P^1 = J.$$

Since $L \simeq K$ (=principal module over $K_t\Gamma$), we see (Corollary 1.3) that the proof of (4)⇒(1) will be completed if we show that the module F is projective over $K_t\Gamma$. To this end let

$$W = \{(h_1 \otimes \cdots \otimes h_n) \in F\colon h_i \in H,\ i = 1, \dots, n\}.$$

This is abuse of notation, but since this set in $K_tH \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} K_tH$ is mapped injectively into F, it should not confuse the reader. The group Γ acts on W, so let $W = W_1 \cup W_2 \cup \cdots \cup W_s$ be the decomposition of W into the Γ-orbits. For each $W_i, i = 1, \dots, s$ let $V_i = \mathrm{span}_K(W_i)$. Clearly, V_i is a $K_t\Gamma$-module and $F = V_1 \oplus \cdots \oplus V_s$ over $K_t\Gamma$. Thus, we must show that each V_i is $K_t\Gamma$ projective. Take $w_i = h_1 \otimes \cdots \otimes h_n \in W_i$ and let $V_i = K_t\Gamma(h_1 \otimes \cdots \otimes h_n)$. We show that the map

$$K_t\Gamma \to V (= V_i),$$
$$\epsilon\colon \sum x_\sigma \sigma \to \sum x_\sigma \sigma(h_1 \otimes \cdots \otimes h_n)$$

splits over $K_t\Gamma$.

Denote $\mathrm{stab}_\Gamma(h_1 \otimes \cdots \otimes h_n)$ by N. A close examination of the action of Γ on $(h_1 \otimes \cdots \otimes h_n)$ reveals that $h(h_1 \otimes \cdots \otimes h_n) = hh_1 \otimes \cdots \otimes \cdots$ for each

$h \in H$ (we assume that $g_1 = 1$). Thus, $h \neq 1$ does not fix $(h_1 \otimes \cdots \otimes h_n)$, i.e., $N \cap H = \langle 1 \rangle$.

By assumption there exists $x_N \in K$ with $\sum_{\tau \in N} \tau(x_N) = 1$.

Define $\varphi: V \to K_t\Gamma$

$$(h_1 \otimes \cdots \otimes h_n) \to \sum_{\tau \in N} \tau(x_N)\tau$$

and extend φ linearly over $K_t\Gamma$. To show that it is well defined let

$$b = \sum_{\sigma \in \Gamma} x_\sigma \sigma (h_1 \otimes \cdots \otimes h_n) = 0.$$

We rewrite it as $\sum_{i=1}^m t_i a_i (h_1 \otimes \cdots \otimes h_n)$, $\{t_1, \ldots, t_m\}$ representatives of the right cosets of N in Γ and $a_i = \sum_{\tau \in N} x_{i,\tau}\tau \in K_t N$. Since $(h_1 \otimes \cdots \otimes h_n)$ is N invariant, we have

$$\sum_{i=1}^m t_i a_i (h_1 \otimes \cdots \otimes h_n) = \sum_{i=1}^m t_i y_i (h_1 \otimes \cdots \otimes h_n), \quad \text{where } y_i = \sum_{\tau \in N} x_{i,\tau}.$$

The n-tuples $\{t_i(h_1 \otimes \cdots \otimes h_n)\}_{i=1}^m$ are distinct and so linearly independent over K. Therefore $y_i = 0$ for $i = 1, \ldots, m$. The elements of N fix $\sum_{\tau \in N} \tau(x_N)\tau$ so

$$a_i \sum_{\tau \in N} \tau(x_N)\tau = y_i \sum_{\tau \in N} \tau(x_N)\tau = 0, \quad \text{i.e., } \varphi(b) = 0.$$

The fact that $\sum_{\tau \in N} \tau(x_N) = 1$ shows that $\epsilon \circ \varphi = \mathrm{id}_V$. This completes the proof of the implication (4)⇒(1) and of Theorem 0.1.

2. How to find x_Γ?

In this section we give an explicit formula for x_Γ, given the elements x_H and x_S for the various S's such that $S \cap H = (1)$.

The way this formula was obtained was by following step by step the proof of implication 4 ⇒ 1. Instead of doing it here we write the expression and prove directly that $\mathrm{tr}_\Gamma(x_\Gamma) = 1$.

Let $H \leq \Gamma$ be any subgroup of Γ. Denote $\mathcal{M} = \{S \leq \Gamma\colon S \cap H = 1\}$. For H and for every $S \in \mathcal{M}$ let x_H, x_S be elements in K with $\mathrm{tr}_H(x_H) = 1$ and $\mathrm{tr}_S(x_S) = 1$. Consider the set of n-tuples $W = \{(h_1, \ldots, h_n)\colon h_i \in H\}$ where

$n = |G\colon H|$. The group Γ acts on the set W as follows: for $(x_1, \ldots, x_n) \in W$, $\sigma \in \Gamma$ put

$$\sigma(x_1, \ldots, x_n) = \left(h_{\nu_1} x_{\nu_1}, \ldots, h_{\nu_n} x_{\nu_n}\right)$$

where $\nu \in \mathrm{Sym}(n)$ and $h_{\nu_i} \in H$ are defined in (1.3.1).

Let $W = W_1 \cup \cdots \cup W_s$ be the decomposition of W into its disjoint orbits. Let $w_i = (h_{i,1}, \ldots, h_{i,n}) \in W_i$ for $i = 1, \ldots, s$ be a representative and let $N_i = \mathrm{stab}_\Gamma(w_i)$. Since $N_i \cap H = \langle 1 \rangle$, there exists $x_{N_i} \in K$ with $\mathrm{tr}_{N_i}(x_{N_i}) = 1$.

THEOREM 2.1: *The element x_Γ given by*

$$x_\Gamma = \sum_{i=1}^{s} x_{N_i} \prod_{j=1}^{n} g_j h_{i,j}(x_H)$$

satisfies $\mathrm{tr}_\Gamma(x_\Gamma) = 1$. Here $(g_1, \ldots, g_n)$ is a set of representatives of the right cosets of H in Γ.

Proof: We prove

$$\sum_{\sigma\in\Gamma} \sigma \sum_{i=1}^{s} \Big[\prod_{j=1}^{n} g_j h_{i,j}(x_H)\Big] x_{N_i} = 1.$$

Denoting the above expression by z we have

$$\begin{aligned} z &= \sum_{\sigma\in\Gamma} \sum_{i=1}^{s} \sigma\Big[\prod_{j=1}^{n} g_j h_{i,j}(x_H)\Big]\sigma(x_{N_i}) = \sum_{i=1}^{s} \sum_{\sigma\in\Gamma} \sigma\Big[\prod_{j=1}^{n} g_j h_{i,j}(x_H)\Big]\sigma(x_{N_i}) \\ &= \sum_{i=1}^{s} \sum_{\mu\in\mathrm{Tr}[\Gamma\colon N_i]} \mu \sum_{\tau\in N_i} \tau\Big[\prod_{j=1}^{n} g_j h_{i,j}(x_H)\Big]\tau(x_{N_i}) \end{aligned}$$

where $\mathrm{Tr}[\Gamma\colon N_i]$ is a set of representatives for the right cosets of N_i in Γ.

Since $N_i = \mathrm{stab}_\Gamma(h_{i,1}, \ldots, h_{i,n})$, the group N_i fixes the product $\big[\prod_{j=1}^{n} g_j h_{i,j}(x_H)\big]$ and since $\sum_{\tau\in N_i} \tau(x_{N_i}) = 1$, we obtain

$$z = \sum_{i=1}^{s} \sum_{\mu\in\mathrm{Tr}[\Gamma\colon N_i]} \mu \prod_{j=1}^{n} g_j h_{i,j}(x_H).$$

Now consider the set of unordered n-tuples

$$A = \left\{\left(\mu g_1 h_{i,1}, \mu g_2 h_{i,2}, \ldots, \mu g_n h_{i,n}\right)\right\}_{\substack{\mu\in\mathrm{Tr}[\Gamma\colon N_i] \\ i=1,\ldots,s.}}$$

The entries of each n-tuple represent the n different cosets of H in Γ. By the definition of the action of Γ on W it follows that all **unordered** n-tuples obtained in this way are different. Hence,

$$\#A = \sum_{i=1}^{s} [\Gamma\colon N_i] = \sum_{i=1}^{s} \#W_i = \#W = \big(\mathrm{ord}(H)\big)^n$$

$=\#$ of unordered n-tuples where each entry represents a different coset of H in Γ.

After reordering the components in z we see that

$$z = \sum_{(h_{s_1},\dots,h_{s_n})\in W} g_1 h_{s_1}(x_H) g_2 h_{s_2}(x_H)\cdots g_n h_{s_n}(x_H) = 1.$$

Remark: The formula for x_Γ holds under weaker conditions, e.g., K non-commutative but x_H from the center Z of K.

3. Global dimension of skew group rings

In [Z] Zhong Yi has obtained results on the global dimension of crossed products $R^*\Gamma$ where R is a Noetherian ring and Γ a finite group. In particular, for skew group rings over commutative (Noetherian) rings, he has the following theorems.

THEOREM 3.1. ([Z] 5.2): *Let K be a commutative, Noetherian ring. Let Γ be a finite group acting on K and let $K_t\Gamma$ be the corresponding skew group ring. Then the following are equivalent:*

1. $\mathrm{gl.\,dim}.K_t\Gamma < \infty$.
2. (a) $\mathrm{gl.\,dim}.K < \infty$,
   (b) *for every maximal ideal M of K with* $\mathrm{char}(K/M) = p > 0$, $(R/M)_t\Gamma_M$ *is semisimple Artinian where* $\Gamma_M = \{g \in \Gamma\colon M^g = M\}$.
3. (a) $\mathrm{gl.\,dim}.K < \infty$,
   (b) *for every maximal ideal M of K with* $\mathrm{char}(K/M) = p > 0$, $\Gamma(M)$ *contains no elements of order p where*

$$\Gamma(M) = \{g \in \Gamma\colon r^g - r \in M, \text{ for all } r \in K\}.$$

THEOREM 3.2. ([Z] 5.4): *Let $K_t\Gamma$ be a skew group ring as in Theorem 3.1 where gl.dim. $K < \infty$. Then*

1. $\mathrm{gl.dim}\, K_t\Gamma < \infty$ *if and only if for all primes p which are not units in K*, $\mathrm{gl.dim}\,.K_t\Gamma_p < \infty$, *where* Γ_p *is any Sylow p-subgroup of* Γ.
2. $\mathrm{gl.dim}\,.K_t\Gamma < \infty$ *if and only if for all primes p which are not units in K*, $\mathrm{gl.dim}\,.K_tP < \infty$, *where P is any elementary abelian p-subgroup of* Γ.

We shall apply Theorem 0.1 in order to generalize Theorem 3.2 to

THEOREM 3.3: *Let* $K_t\Gamma$ *be a skew group ring where K is any commutative ring with finite global dimension. Then* $\mathrm{gl.dim}\,.K_t\Gamma < \infty$ *if and only if for all primes p*, $\mathrm{gl.dim}\,.K_tP < \infty$, *where P is any (cyclic) subgroup of order p.*

Proof: This follows from the applications of Corollaries 0.2, 1.3 and the proposition below for the skew group rings $K_t\Gamma$ and K_tP, $|P| = $ prime.

PROPOSITION 3.4: *Let K be a commutative ring acted on by a finite group* Γ. *Then* $\mathrm{gl.dim}\,.K_t\Gamma < \infty$ *if and only if*

(a) $\mathrm{gl.dim}\,.K < \infty$,

(b) *K is projective as a principal* $K_t\Gamma$ *module.*

Note that Remark 6 above shows that Theorem 3.3 does not extend to non-commutative coefficient rings.

Proof: (See [A1], p. 53, implications (1)⇒ (3) ⇒ (2).)

First note that every K module N is a direct summand of the K module $K_t\Gamma \otimes_K N$ (the action via the left component). Hence,

$$\mathrm{proj.dim}_K N \le \mathrm{proj.dim}_K \left(K_t\Gamma \otimes_K N\right).$$

Now $L = K_t\Gamma \otimes_K N$ has a $K_t\Gamma$ left structure and any projective resolution of L over $K_t\Gamma$ is also a projective resolution of L over K. Thus,

$$\mathrm{proj.dim}_K \left(K_t\Gamma \otimes_K N\right) \le \mathrm{proj.dim}_{K_t\Gamma} \left(K_t\Gamma \otimes_K N\right)$$

showing the necessity of (a).

To show the necessity of condition (b) assume $\mathrm{proj.dim}_{K_t\Gamma} K = r > 0$. Using the right exactness of the functor $\mathrm{Ext}^r_{K_t\Gamma}(K, -)$ and Theorems 9.1, 9.2 in [S] we get an epimorphism

$$\mathrm{Ext}^r_K(K, N) \simeq \mathrm{Ext}^r_{K_t\Gamma}\left(K, \mathrm{Hom}_K(K_t\Gamma, N)\right) \to \mathrm{Ext}^r_{K_t\Gamma}(K, N) \to 0$$

where N is an arbitrary left $K_t\Gamma$ module. Thus, K is $K_t\Gamma$ projective.

To show the sufficiency of conditions (a) and (b) let M be a $K_t\Gamma$ module and

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

a projective resolution over $K_t\Gamma$. Since projectives over $K_t\Gamma$ are projective over K we get $\operatorname{proj.dim}_{K_t\Gamma} M \leq \operatorname{proj.dim}_K M$ if we show that every $K_t\Gamma$ module C, that is projective over K, is also projective over $K_t\Gamma$. This follows from the natural isomorphism of functors ([B], Lemma 9.3)

$$\operatorname{Hom}_{K_t\Gamma}(C, -) \simeq \operatorname{Hom}_{K_t\Gamma}\big(K, \operatorname{Hom}_K(C, -)\big)$$

and the fact that K is $K_t\Gamma$-projective.

This completes the proof of Proposition 3.4 and of Theorem 3.3.

Finally we combine Theorem 3.1 with Theorem 3.3 to obtain the following:

PROPOSITION 3.5: *Let K be a commutative, Noetherian ring of finite global dimension. Let Γ be a finite group acting on K. For each $P \leq \Gamma$, $|P| =$ prime, let $\mathcal{M}_p \subset K$ be the P-ideal*

$$\mathcal{M}_p = \langle \sigma(x) - x \colon \sigma \in P,\ x \in K \rangle.$$

Denote by $S = \{P \leq \Gamma, \mathcal{M}_p \neq K\}$. Then $\operatorname{gl.dim} K_t\Gamma < \infty$ if and only if for every $P \in S$, $|P| = p \in (K/\mathcal{M}_p)^*$.

Proof: By Theorem 3.3 it is enough to show $\operatorname{gl.dim} K_tP < \infty$ if and only if either $\mathcal{M}_p = K$ or $|P| = p \in (K/\mathcal{M}_p)^*$ where P is any subgroup of Γ of prime order. Assume $\operatorname{gl.dim} K_tP < \infty$ and $\mathcal{M}_p \neq K$. Note that P acts on $K/\mathcal{M}_p$ trivially, therefore $(K/\mathcal{M}_p)_t P \simeq (K/\mathcal{M}_p) P$. If p is not invertible in $K/\mathcal{M}_p$, then $p \equiv 0$ in K/I for some maximal ideal I containing $\mathcal{M}_p$. Clearly, $(K/I)P$ is not semisimple Artinian which is a contradiction to Theorem 3.1 (1)$\Rightarrow$(2). To show sufficiency, let I be a maximal ideal in K and let $P_I = \{\sigma \in P \colon I^\sigma = I\}$. If $I \not\supset \mathcal{M}_p$ then either P acts *nontrivially* on K/I or $P_I = <1>$. In both cases $(K/I)_tP_I$ is semisimple Artinian. If $I \supset \mathcal{M}_p$ then $P \in$S. Hence, p is invertible in $K/\mathcal{M}_p$ and therefore in K/I. Clearly, $(K/I)_tP_I$ $(= (K/I)P)$ is semisimple Artinian. Now use Theorem 3.1 (2)$\Rightarrow$(1).

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